Categories of abstract and noncommutative measurable spaces

Joint work with Tobias Fritz

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Preliminaries and motivation

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(**Quantum logic:** ortholattices).

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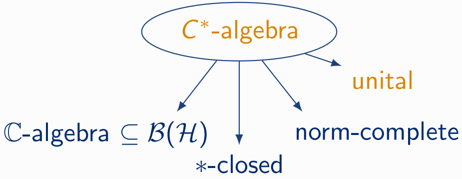

Positive cone: $a \geq 0 \overset{\text{def}}{\iff} a = b^* b$ for some b.

Main examples

- M_n, the algebra of $n \times n$ matrices. For $A \in M_n$,

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Gelfand duality

$X \mapsto C(X)$ yields a contravariant equivalence

$$\left\{\begin{array}{l}\text{compact Hausdorff}\\ \text{spaces and continuous}\\ \text{maps}\end{array}\right\} \xrightarrow{\cong^{\mathrm{op}}} \left\{\begin{array}{l}\text{commutative } C^*\text{-}\\ \text{algebras and } *\text{-}\\ \text{homomorphisms}\end{array}\right\}$$

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Probabilistic Gelfand duality[1]

$$\left\{\begin{array}{l}\text{compact Hausdorff spaces} \\ \text{and } \textit{continuous} \text{ Markov} \\ \text{kernels}\end{array}\right\} \xrightarrow{\cong^{\mathrm{op}}} \left\{\begin{array}{l}\text{commutative } C^*\text{-} \\ \text{algebras and cpu maps}\end{array}\right\}$$

[1] Robert W. J. Furber and Bart P. F. Jacobs. From Kleisli categories to commutative C*-algebras: Probabilistic Gelfand duality. *Logical Methods in Computer Science*, 11, 2015.

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Disclaimer[2]

Similar goal: find an equivalence with von Neumann algebras, but then we have a less familiar commutative side (enhanced measurable spaces).

[2] D. Pavlov. Gelfand-type duality for commutative von Neumann algebras. *J. Pure Appl. Algebra*, 226(4):106884, 2022.

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Deterministic side

The C^*-algebra associated to a measurable space

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$\Longrightarrow$ ~~Equivalence~~ Adjunction?

What additional property is satisfied by $\mathcal{L}^\infty(X)$?

[3]Kazuyuki Saitô and J. D. Maitland Wright. *Monotone Complete C*-algebras and Generic Dynamics*. Springer Monogr. Math. Springer London, 2015.

σC^*-algebras

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A σC^*-algebra $\mathcal{A}$, or monotone σ-complete C^*-algebra[3], is a C^*-algebra such that (every bounded monotone increasing sequence (a_n) has a supremum $\sup_n a_n$)

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Instructive example

$$L^\infty([0,1]) := \left\{ \begin{array}{l} \lambda\text{-}\textit{a.s. classes} \text{ of bounded measurable} \\ \text{functions } f\colon [0,1] \to \mathbb{C} \end{array} \right\}$$

is a commutative σC^*-algebra such that $L^\infty([0,1]) \ncong \mathcal{L}^\infty(X)$ for any choice of measurable X.

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Adjunction with measurable spaces

Measurable Gelfand duality

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σ-homomorphism: $*$-homomorphism that preserves the suprema of bounded monotone increasing sequences.

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Examples

$\mathcal{B}(\mathbb{C})$ the Borel σ-algebra of $\mathbb{C}$, $\mathcal{B}(\mathbb{C})/M$ and $\mathcal{B}(\mathbb{C})/N$, with M the ideal of meager sets and N the ideal of Lebesgue null sets.

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NB: $L^\infty([0,1]) \cong \mathcal{L}^\infty_{\mathrm{abs}}(\mathcal{B}([0,1])/N)$.

Equivalence with Boolean σ-algebras

Theorem

$$\left\{\begin{array}{l}\text{Boolean } \sigma\text{-algebras and} \\ \sigma\text{-homomorphisms}\end{array}\right\} \underset{\text{Proj}}{\overset{\mathcal{L}^\infty_{\text{abs}}}{\rightleftarrows}} \cong \left\{\begin{array}{l}\text{commutative } \sigma C^*\text{-} \\ \text{algebras and } \sigma\text{-} \\ \text{homomorphisms}\end{array}\right\}$$

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In particular, $\mathcal{L}^\infty(X) \cong \mathcal{L}^\infty_{\text{abs}}(\Sigma_X)$.

NB. The adjunction between Boolean σ-algebras and measurable spaces was already noted.[4]

[4] Ruiyuan Chen. A universal characterization of standard Borel spaces. *J. Symb. Log.*, 88(2):510–539, 2023.

Restriction to meaningful measurable spaces

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Pedersen–Baire envelopes

For $\mathcal{A}$ a C^*-algebra, there exists a σC^*-algebra $\mathcal{A}^\infty$, called the Pedersen–Baire envelope of $\mathcal{A}$, such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\text{hom}"] \arrow[d, hook] & \mathcal{B} \\ \mathcal{A}^\infty \arrow[ur, dashed, "! \, \sigma\text{hom}"'] & \end{tikzcd}$$

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for any σC^*-algebra $\mathcal{B}$.

In other words, $(-)^\infty$ is a functor from C^*-algebras to σC^*-algebras, left adjoint to the forgetful functor.

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NB. Baire measurable spaces have been investigated as a good framework outside "countable restrictions".[5]

[5] Asgar Jamneshan and Terence Tao. Foundational aspects of uncountable measure theory: Gelfand duality, Riesz representation, canonical models, and canonical disintegration. Fundam. Math., 261(1):1–98, 2023.

Standard Borel spaces

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- commutative Pedersen–Baire envelopes of separable C^*-algebras, and
- countably presented Boolean σ-algebras.[4]

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Most used one

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Probabilistic side

Probability measures and generalized POVMs

Let B be a Boolean σ-algebra. A *probability measure* on B is a function $\mu\colon B \to \mathbb{C}$ such that

1. (positivity) $\mu(p) \geq 0$ for all $p \in B$;
2. (normalization) $\mu(\top) = 1$;
3. (σ-additivity) (p_n) countable sequence pairwise disjoint,

$$\mu\left(\bigvee_n p_n\right) = \sum_n \mu(p_n)$$

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NB. If $B = \Sigma_Y$ and $\mathcal{A} = \mathcal{L}^\infty(X)$, this is basically a Markov kernel $X \to Y$!

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Theorem

Let $\mu \colon B \to \mathcal{A}$ a POVM. Then

$$\begin{tikzcd} B \arrow[r, "\mu"] \arrow[d, "\chi_-"'] & \mathcal{A} \\ \mathcal{L}^\infty_{\mathsf{abs}}(B) \arrow[ur, "\tilde{\mu}"'] & \end{tikzcd}$$

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Idea: Lebesgue integral![6]

[6] J. D. Maitland Wright. Measures with Values in a Partially Ordered Vector Space. Proceedings of the London Mathematical Society, s3-25(4):675–688, 11 1972.

POVMs on lattices?

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A possible way to overcome this issue is by considering unitary actions,[7] but this is beyond our scope.

[7] C. Heunen and M. L. Reyes. Active lattices determine AW^*-algebras. J. Math. Anal. Appl., 416(1):289– 313, 2014.

Final results

Corollary

$$\left\{ \begin{array}{l} \text{measurable spaces and} \\ \text{Markov kernels} \end{array} \right\} \overset{f.f.}{\hookrightarrow}{}^{\mathrm{op}} \left\{ \begin{array}{l} \text{commutative } \sigma C^*\text{-} \\ \text{algebras and } \sigma\text{-cpu} \\ \text{maps} \end{array} \right\}$$

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Corollary

$$\left\{ \text{Baire measurable spaces and Markov kernels} \right\} \xrightarrow{\cong^{\mathrm{op}}} \left\{ \text{commutative Pedersen–Baire envelopes and } \sigma\text{-cpu maps} \right\}$$

...and restriction to standard Borel spaces

Corollary

$$\left\{\begin{array}{l}\text{standard Borel spaces}\\ \text{and Markov kernels}\end{array}\right\} \xrightarrow{\cong^{\mathrm{op}}} \left\{\begin{array}{l}\text{commutative Pedersen–}\\ \text{Baire envelopes of}\\ \text{separable } C^*\text{-algebras}\\ \text{and } \sigma\text{-cpu maps}\end{array}\right\}$$

Two remarks

- We can equip Pedersen–Baire envelopes with a tensor product giving rise to a symmetric monoidal category (whose monoidal unit is initial);

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- The last two equivalences are strong monoidal functors (they also preserve the Markov category structures).

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- Algebraic descriptions of measurable spaces (σC^*-algebras and Boolean σ-algebras);
- Stronger connections to quantum logic (σ-complete ortholattices);
- Markov kernels are σ-cpu maps;
- Extension of Markov kernels to Boolean σ-algebras;
- Plausible setting for measurable studies in quantum probability (Pedersen–Baire envelopes).

THANK YOU FOR YOUR ATTENTION